Use of notations $n!$ and $\binom{n}{r}$.

Factorial

Dr. K.M. Hock

$3!$ means $3 \times 2 \times 1$
$5!$ means $5 \times 4 \times 3 \times 2 \times 1$, etc.

$n! = n(n-1)(n-2)\ldots \times 2 \times 1$.

$0! = ?$ define $0! = 1$.

Use.

e.g. 3 letters a b c. How many ways to arrange in a row?

_ _ _

1st place → 3 ways - a, b or c, e.g. b

2nd place → 2 ways left, e.g a

3rd place → 1 way left. e.g. c.

$\therefore$ $3 \times 2 \times 1$ ways to arrange a, b, c.

So $3!$ = no. of ways to arrange 3 different things in a row

→ $n!$ = no. of ways to arrange n different things in a row

Use of notations $n!$ and $\binom{n}{r}$.

Arrangements

Dr. K.M. Hock

e.g. a, b, c, d, e.
How many ways to pick 3 and arrange in a row?

$$\_\_ \quad \_\_ \quad \_\_$$

first place: 5 ways - a, b, c, d, e. e.g. b
2nd place: 4 ways left - a, c, d, e only. e.g. e
3rd place: 3 ways left. - a, c, d.

$\therefore$ no. of ways $= 5 \times 4 \times 3$

Can also write as $\dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \dfrac{5!}{2!}$

or $\dfrac{\overset{5 \text{ things}}{5!}}{\underbrace{(5-3)!}_{\text{pick } 3}}$

$\rightarrow$ So if n things pick r,
then no. of ways to arrange in row $= \dfrac{n!}{(n-r)!}$

Use of notations $n!$ and $\binom{n}{r}$.

Combination

Dr. K.M. Hock

e.g a, b, c, d, e.
How many ways to pick 3 if I am not interested in arranging in a row?

From last page, if arrange in row,
no. of ways $= \frac{5!}{(5-3)!}$

For each combination picked, like $a\ c\ d$, $3!$ ways to arrange in a row.

So no. of combinations possible $\times\ 3! = \frac{5!}{(5-3)!}$

$\therefore$ no. of combinations $= \frac{5!}{(5-3)!\,3!}$

→ If from n things I pick r
(and I don't care about their arrangements),
no. of combinations $= \frac{n!}{(n-r)!\,r!}$

Notation $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

Can also write as $\binom{n}{r} = \frac{n(n-1)\ \cdots\ (n-r+1)}{r!}$

Expansion

Dr. K. M. Hock

e.g

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Is there a pattern? A formula?

---

e.g $(a+b)(a+b)(a+b)$

Way To expand
- pick one term from each bracket
- multiply these terms
- repeat for all combinations
- add

Can have
a^3 - 1 way to pick
a^2b - 3 ways to pick b
ab^2 - pick 2 b from 3 brackets? $\binom{3}{2} = 3$
b^3 - 1 way to pick

So $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

---

e.g $(a+b)(a+b)(a+b)(a+b)$

a^4 - pick 0 b from 4 brackets? $\binom{4}{0} = 1$
a^3b - pick 1 b from 4 brackets? $\binom{4}{1} = 4$
a^2b^2 - pick 2 b from 4 brackets? $\binom{4}{2} = 6$
ab^3 - pick 3 b from 4 brackets? $\binom{4}{3} = 4$
b^4 - pick 4 b from 4 brackets? $\binom{4}{4} = 1$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Binomial Theorem

Dr. K. M. Hock

e.g. Previously,

$$(a+b)^4 = \binom{4}{0} a^4 b^0 + \binom{4}{1} a^3 b^1 + \binom{4}{2} a^2 b^2 + \binom{4}{3} a^1 b^3 + \binom{4}{4} a^0 b^4$$

→ Binomial theorem

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-1} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n.$$

note: $\binom{n}{1} = n$. $\binom{n}{0} = 1$.

E.g. Find the 4th term of $(a+b)^{10}$.

Let $n = 10$, $r = 4$.

$$\binom{n}{r} a^{n-r} b^r = \binom{10}{4} a^{10-4} b^4$$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} a^6 b^4$$

$$= 210\, a^6 b^4$$

Problem

Dr. K. M. Hock

2013 P1 Q5 The coefficient of x^3 in the expansion of $(a-x)^5 + (2+x)^6$ is 70.

(i) Find the value of the positive constant a.

(ii) Hence calculate the coefficient of x^2 in the expansion of $(a-x)^5 + (2+x)^6$.

Solution. (i)

$(a-x)^5$: coeff. of x^3

$$= \binom{5}{3} a^{5-3}(-x)^3$$
$$= -\frac{5\times4\times3}{3\times2\times1} a^2 x^3$$
$$= 10 a^2 x^3$$

$(2+x)^6$: coeff. of x^3

$$= \binom{6}{3} 2^{6-3} x^3$$
$$= \frac{6\times5\times4}{3\times2\times1} 2^3 x^3$$
$$= 160 x^3$$

Total is $10a^2x^3 + 160x^3 = (10a^2 + 160)x^3$

Given that coeff. of x^3 is 70.

So $10a^2 + 160 = 70 \Rightarrow a^2 = 9 \Rightarrow a = 3$

since a positive.

(ii) $(3-x)^5 + (2+x)^6$

x^2 term :

$$\binom{5}{2} 3^{5-2}(-x)^2 + \binom{6}{2} 2^{6-2} x^2$$
$$= \frac{5\times4}{2\times1} \times 3^3 x^2 + \frac{6\times5}{2\times1} \times 2^4 x^2$$
$$= 270x^2 + 240x^2$$

Coeff. of $x^2 = 270 + 240 = 510$